Numerical Modeling of the Atmosphere: A Review

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Plan

- Underlying principles and related issues
- Various approaches to numerical modeling
- Representation of spherical Earth
- Representation in vertical and topography
- Non-hydrostatic models and NGGPS (Next Generation Global Prediction System)
- Emerging methods

(Loosely following paper published in Oxford Research Encyclopedia of Climate Science by Fedor Mesinger, Miodrag Rancic and Robert James Purser)

Underlying Principles

- This talk will try to describe some of important elements of numerical methods used in formulation of numerical models of atmospheric dynamics
- More complete reviews of the subject can be found in textbooks, such as:

Kalnay, E. (2003). *Atmospheric Modeling, Data Assimilation and Predictability*. Cambridge Univ. Press, 320 pp

Durran, D. R. (2010). Numerical Methods for Fluid Dynamics, with Applications to Geophysics. Series: Texts in Applied Mathematics, Vol. 32, 2nd Edition., 2010, XV, 516 pp.

as well as in the review articles, for example:

Cullen, M. J. P. (2007). Modelling atmospheric flows. *Acta Numerica*, 1-87. Cambridge Univ. Press.

Williamson, D. L. (2007). The evolution of dynamical cores for global atmospheric models. *Journal of the Meteorological Society of Japan*, 85B: 241–269.

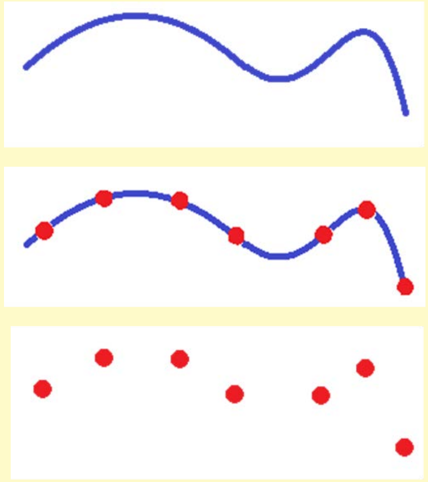
Côté, J., Jablonowski, C., Bauer, P., & Wedi, N. (2015). Numerical methods of the atmosphere and ocean. In *Seamless prediction of the Earth system: From minutes to months* (pp. 101–124). World Meteorological Organization, WMO-No. 1156.

Lauritzen, P. H., Jablonowski, C., Taylor, M. A., & Nair, R. D., Eds. (2011). *Numerical Techniques for Global Atmospheric Models*, Lecture Notes in Computational Science and Engineering, Springer, Vol. 80.

Initial value problem

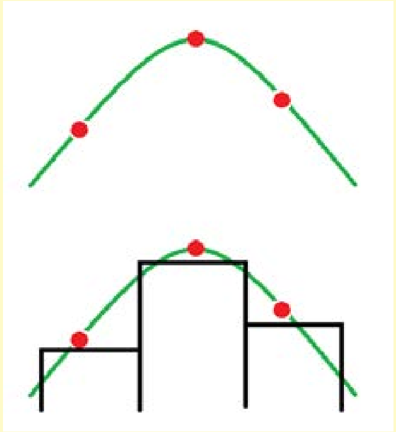
Given an initial field of the state variables (e.g., pressure, temperature, winds, …), and knowing equations that govern their evolution in time, we can, in principle, find their future state (Vilhelm Bjerknes, 1904)

Problem 1: We do not exactly know fields except at certain points in space, which means that we deal with finite degrees of freedom. Our continuous equations, even if we were able to solve them analytically in the general case (we do not!) are not applicable for discrete case, and we need to use their approximations and solve them using approximate methods.



Finite degrees of freedom, that is, a **discrete knowledge** of the fields

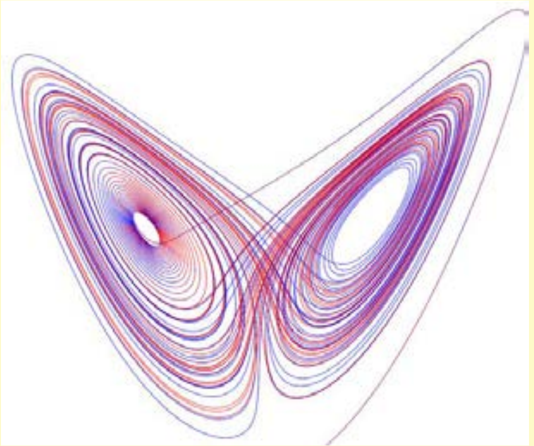
At the same time we need to use "representative" rather then "instantaneous values" of variables. For example, the famous first numerical forecast attempt by Lewis Fry Richardson (1922) was in a later reconstruction surprisingly successful only by including spatial filtering of the initial fields



Instantaneous and **representative** values of the fields

Problem 2: Physical constraints

Equations that govern atmospheric dynamics represent mathematical expressions of physical laws and have a series of physical constraints that a realistic solution of our approximate equations needs to reflect (e.g., conservation of mass, rotational properties, energy, positive definiteness of tracers, etc., sometimes referred to as mimetic properties)



Problem 3: Nonlinearity

Our equations are "nonlinear", and our task all but hopeless, because, as pointed out by Edward Lorenz (1960), small perturbations in initial conditions can lead to qualitatively different solutions

Problem 4: Numerical instabilities

Numerical solutions require that certain conditions are satisfied in order to avoid computational instabilities. A typical example is Courant-Friedrichs-Lewy (or linear stability condition): $\Delta t < \frac{\Delta x}{c}$

Norman Phillips (1956) discovered different, nonlinear computational instability, which he later explained in his (1959) paper as an erroneous accumulation of energy at the shortest scales. That led Akio Arakawa (1966) to devising an advection scheme that by conserving chosen integral properties of the continuous equations eliminated the problem, and which became a forerunner of later efforts in emulating various properties of the physical system.

Problem 5: Numerical techniques suffer from inadequate dispersion of the short waves and may generate computational noise.

Short waves generally are not well represented in numerical solutions, as shown on the dispersion diagram of gravity inertial waves

(Note that contribution from physics forcing generally happens in the short portion of the spectrum, and that the gravity waves are partially responsible for spreading this signal to surroundings toward reestablishing the geostrophic balance)

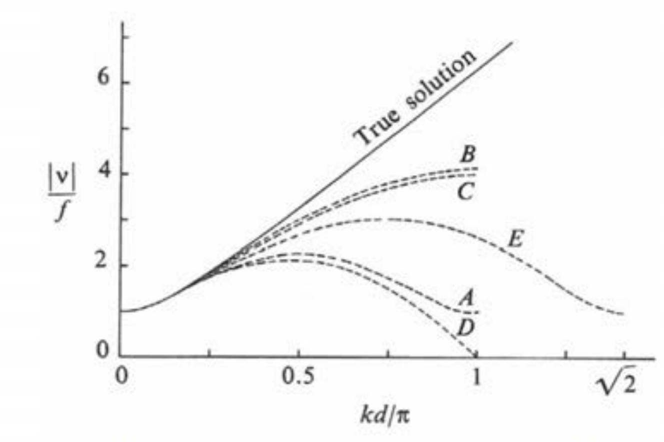


Figure 3.2 The functions |v|/f given by (3.4) and (3.6), with $\lambda/d = 2$.

Typically, high-order approximations to the first derivative produce, as a nus result, a false, computational solution. Examples are 4-th order spatial schemes and three-time-level schemes.

Problem 6: How to deal with "subgrid scales"?

Models are using different mechanisms to describe this subscale dissipation, such as horizontal diffusion, divergence damping, smoothers, filters and fixers, as neatly described in a review of the subject by

Jablonowski, C., & Williamson, D. L. (2011). The pros and cons of diffusion, filters and fixers in atmospheric general circulation models. In P. H. Lauritzen et al., (Eds.), *Numerical Techniques for Global Atmospheric Models*, LNCSE Vol. 80 (pp. 381– 493). Springer-Verlag.

There is no universal theory how to deal with this issue, though a limited guidance come from Smagorinsky (1993)

An objective criterium of how successfully model deal with subgrid dissipation was suggested by Skamarock (2004, 2011), who introduces a model's "effective resolution" as the one at which kinetic energy spectrum (KE) starts to be steeper then $k^{-5/3}$ observed in atmosphere for higher wave numbers

For models that ever came close to following this law, that is between 6 and 10 Δx

From the good side:

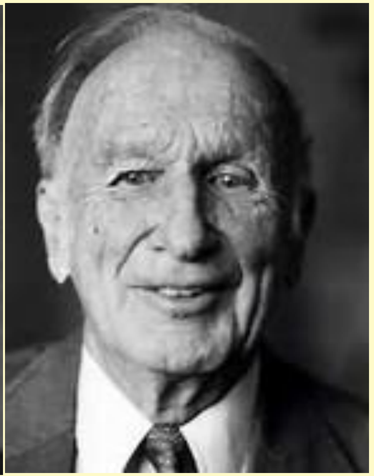
We always have verification for our calculations

Not really!!

RTMA (Real Time Mesoscale Analysis) Project at EMC is able to reconstruct some of weather elements in 15 min intervals only at grids at 1.25 to 2.5 km.

Vilhelm Bjerknes

Lewis Fry Richardson



Edward Lorenz

Norman A. Phillips

Akio Arakawa

Various approaches to numerical modeling

We will focus in this lecture on 3 main groups of methods used in numerical models

- Spectral methods (global models)
- Finite-differencing (regional models)
  (They both belong to Eulerian methods)
- Semi-Lagrangian

Many other techniques has been developed and used in numerical models of the atmosphere (finite-elements, pseudo-spectral, and in recent years, finite-volume, and spectral element methods)

With the advent of massively parallel computers approaching order of 10,000 processors, methods based on global operations, such as spectral, slowly but certainly lose the ground, and the methods based on local calculations are becoming more interesting again.

Criteria in contemporary numerical models

Staniforth and Thuburn (2012) suggested a list of "essential, or at least highly desirable" properties that model dynamical cores should have:

1. Mass conservation;
2. Accurate representation of balanced flow and adjustment;
3. Computational modes should be absent or well controlled;
4. The geopotential gradient and pressure gradient should produce no unphysical source of vorticity;
5. Terms involving the pressure should be energy conserving;
6. Coriolis terms should be energy conserving;
7. There should be no spurious fast propagation of Rossby modes; geostrophic balance should not spontaneously break down;
8. Axial angular momentum should be conserved;
9. Accuracy approaching second order; (Or higher ???? How about seasonal predictions?)
10. Minimal grid imprinting.

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Spectral methods

- Spectral methods dominated global modeling and were almost exclusively used as GCM (GFS, ECMWF, ARPEGE, …) until advent of massively parallel computers.
- Extensive reviews of spectral approach can be found in

Machenhauer, B., 1979: The spectral method. GARP Publication Series No. 17, Vol. II, 124-275.

Machenhauer, B., 1991: Spectral methods. *ECMWF Seminar Proceedings: Numerical Methods in Atmospheric Models, Vol. I*. ECMWF, Reading, United Kingdom, 3-85.

Terry Davies, 2002: Adiabatic formulation of models. Meteorological Training Course Lecture Notes, ECMWF [Available online]

Williamson, D. L., and R. Laprise, 2000: Numerical approximations for global atmospheric general circulation models. In: Numerical Modeling of the Global Atmosphere in the Climate System, P. Mote and A. O'Neill (Eds.), Kluwer, 127-219

- Spectral models use as basis functions spherical harmonics, which represent solution of the Laplace equation on the sphere

$$\nabla^2 Y_n^m = \frac{1}{a^2}\left[\frac{1}{\cos^2\varphi}\frac{\partial^2 Y_n^m}{\partial\lambda^2} + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial Y_n^m}{\partial\varphi}\right)\right]$$

$$= \frac{-n(n+1)}{a^2} Y_n^m \tag{3.3.21}$$

The spherical harmonics are products of Fourier series in longitude and associated Legendre polynomials in latitude:

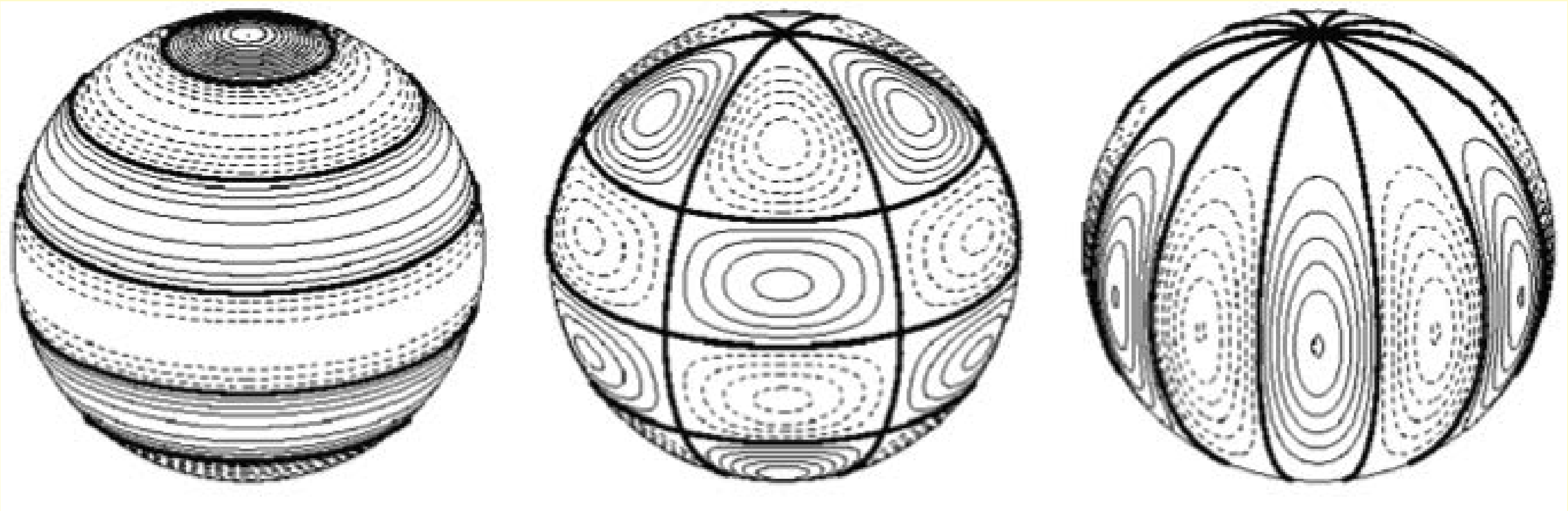
$$Y_n^m(\lambda,\varphi) = P_n^m(\mu)e^{im\lambda} \tag{3.3.22}$$

where $\mu = \sin\varphi$, m is the zonal wavenumber and n is the "total" wavenumber in spherical coordinates (as suggested by the Laplace equation (3.3.21)). P_n^m are the associated Legendre polynomials in $\mu = \sin\varphi = \cos\theta$, where $\theta = \pi/2 - \varphi$ is the co-latitude. For example, the $P_0^0 = 1$; $P_1^0 = \cos\theta$; $P_1^1 = \sin\theta$; $P_2^0 = 1/2\,(3\cos^2\theta - 1)$; $P_2^1 = 3\sin\theta\cos\theta$; $P_2^1 = 3\sin^2\theta$; ...

Using "triangular" truncation

$$U(\lambda,\varphi,t) = \sum_{n=0}^{N}\sum_{m=-n}^{n} U_n^m(t)Y_n^m(\lambda,\varphi) \tag{3.3.23}$$

the spatial resolution is uniform throughout the sphere. This is a major advantage



Williamson and Laprise (1998). (a) Depiction of three spherical harmonics with total wavenumber $n = 6$. Left, zonal wavenumber $m = 0$; center, $m = 3$; right, $m = 6$. Note that n is associated with the total wavelength (twice the distance between a maximum and a minimum), which is the same for the three figures.

- Application of FFT and spectral transform for calculation of nonlinear terms made them very efficient
- Technically "indefinite" order of accuracy
- Conservation of quadratic quantities (energy, enstrophy)
- Semi-Lagrangian methods for horizontal advection (ECMWF, GFS)

Finite-differencing methods

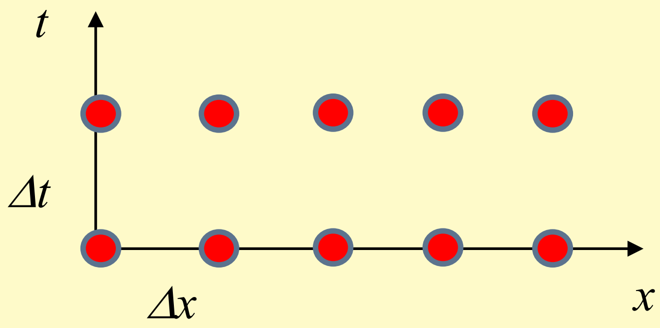
- Finite-differencing methods were historically reserved for high-resolution limited-area (regional) models, which use the time evolution of the solution at the boundaries ("boundary conditions") from global models

- The continuous derivatives are approximated with a consistent quotients of finite-differences, for example:

$$\frac{\partial u}{\partial x} \to \frac{\Delta u}{\Delta x} \quad ; \quad \frac{\partial u}{\partial t} \to \frac{\Delta u}{\Delta t}$$

where Δx and Δt are grid increments in space and time, respectively.

- Using Taylor series formula, one can find the order of accuracy of finite-differencing approximations



Schemes with centered fourth-order space differencing

$$\frac{u_{j+1}-u_{j-1}}{2\Delta x}=\frac{\partial u}{\partial x}+\frac{1}{3!}\frac{\partial^3 u}{\partial x^3}(\Delta x)^2+0\left[(\Delta x)^4\right]. \tag{4.1}$$

Thus, this quotient is of the second order of accuracy. It is formed by taking differences of values of u_j at points one grid distance away from the central point. Similarly a quotient can be formed by taking differences of values two grid distance away. We then obtain, replacing Δx in (4.1) by $2\Delta x$,

$$\frac{u_{j+2}-u_{j-2}}{4\Delta x}=\frac{\partial u}{\partial x}+\frac{4}{3!}\frac{\partial^3 u}{\partial x^3}(\Delta x)^2+0\left[(\Delta x)^4\right]. \tag{4.2}$$

This quotient is still second order accurate, but the coefficients are larger. Other consistent approximations to $\partial u/\partial x$ can be formed as linear combinations of the quotients (4.1) and (4.2). The combination for the second order terms in the truncation errors of (4.1) and (4.2) to cancel is particularly important. This is

$$\frac{4}{3}\frac{u_{j+1}-u_{j-1}}{2\Delta x}-\frac{1}{3}\frac{u_{j+2}-u_{j-2}}{4\Delta x}=\frac{\partial u}{\partial x}+0\left[(\Delta x)^4\right], \tag{4.3}$$

- For example, when solving a linear advection equation

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

in analytical solution the initial state propagates in space at constant phase speed c without change of shape. The numerical solution however will leg behind analytical and disperse. The higher the order of accuracy, the smaller phase lag:

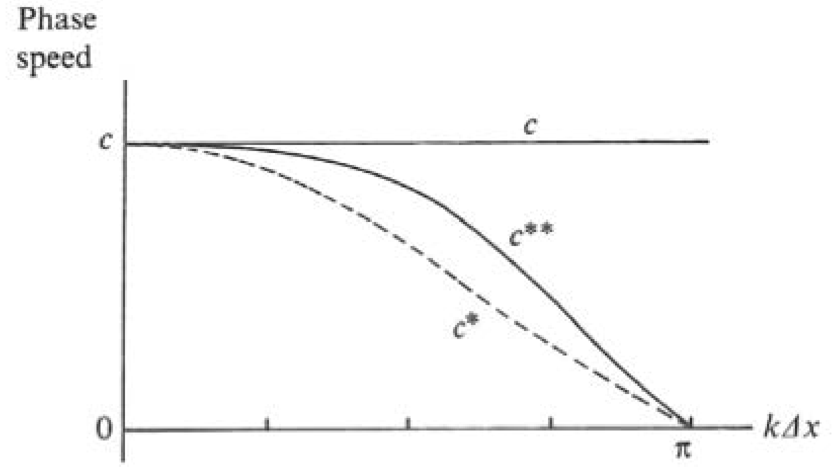
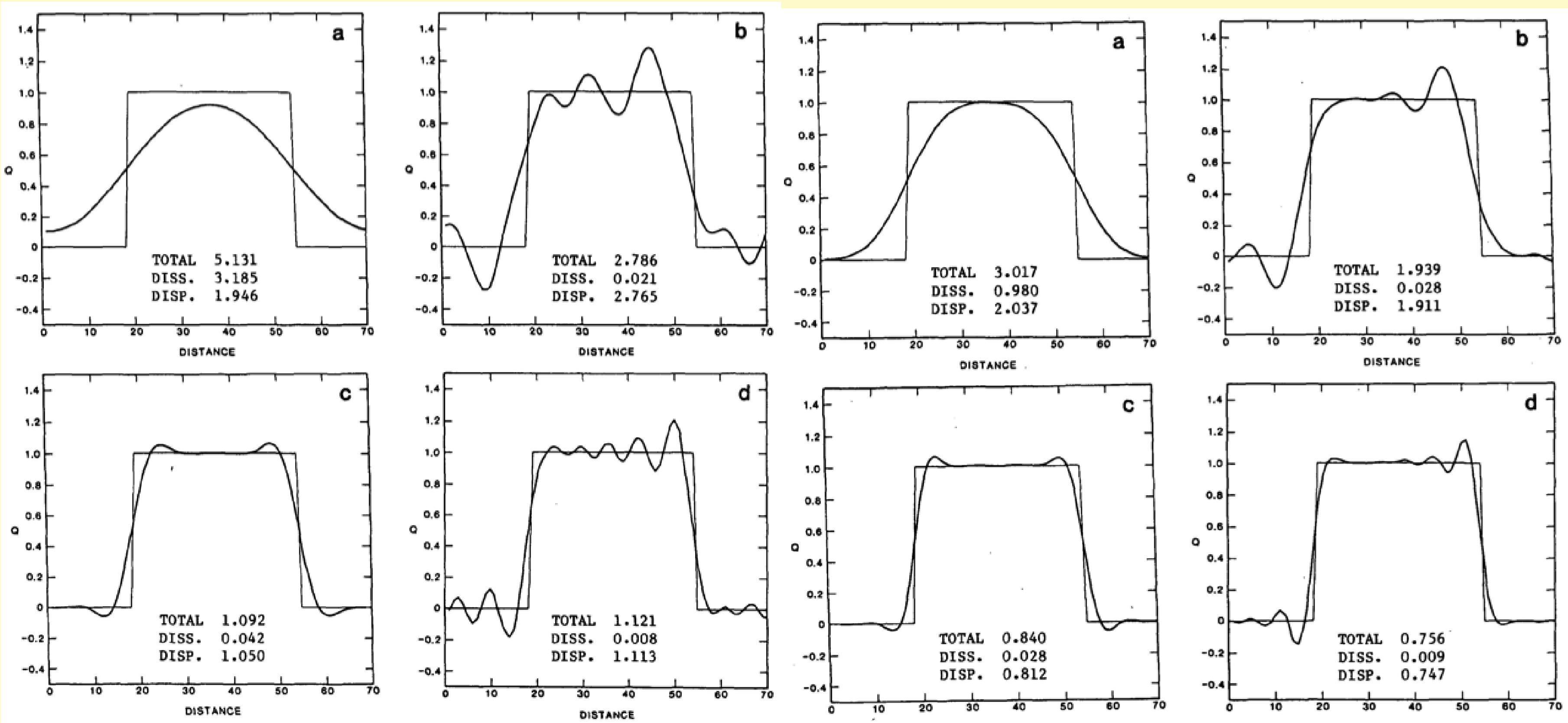


Figure 4.1 Phase speed for the linear advection equation, c, and for the corresponding differential-difference equations with second order (c^*) and with fourth order (c^{**}) centered space differencing.

An increase of order of accuracy does not solve the problem because of dispersion that numerical solution involves (From Takacs, 1985):

$m = 0.2$ $m = 0.7$



Numerical solutions for (a) 1st-order scheme, (b) 2nd-order scheme, (c) 3rd-order scheme, and (d) 4th-order scheme, after two complete translations

Treatment of nonlinearity

In the case of nonlinear advection $\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$

the nonlinear term is leading to a so-called "aliasing error". Assume that u= $\sin(kx)$
In that case,

$$u\frac{\partial u}{\partial x} = kcos(kx)\sin(kx) = \frac{k}{2}\sin(2kx)$$

with the effect that we get a new wave whose wavenumber (2k) is twice that of the initial wave number (k). The maximum wave number that grid can recognize is

$$k_{max} = \frac{2\pi}{2\Delta x} = \frac{\pi}{\Delta x}$$

If the new wave is larger than the maximum value, the grid will not be able to recognize it, and instead, it will show it as

$$k^* = 2k_{max} - k$$

$$\sin kx = \sin\left[2k_{max} - (2k_{max} - k)\right]x$$

$(\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta)$

In general:

$$\sin kx = \sin\frac{2\pi}{\Delta x}x \cos\left(\frac{2\pi}{\Delta x} - k\right)x - \cos\frac{2\pi}{\Delta x}x \sin\left(\frac{2\pi}{\Delta x} - k\right)x .$$

However, at the grid points $x = j\Delta x$, and

$$\sin\frac{2\pi}{\Delta x}j\Delta x = 0, \quad \cos\frac{2\pi}{\Delta x}j\Delta x = 1 .$$

Therefore, we find

$$\sin k\, j\Delta x = -\sin\left(2k_{max} - k\right)j\Delta x . \qquad (6.3)$$

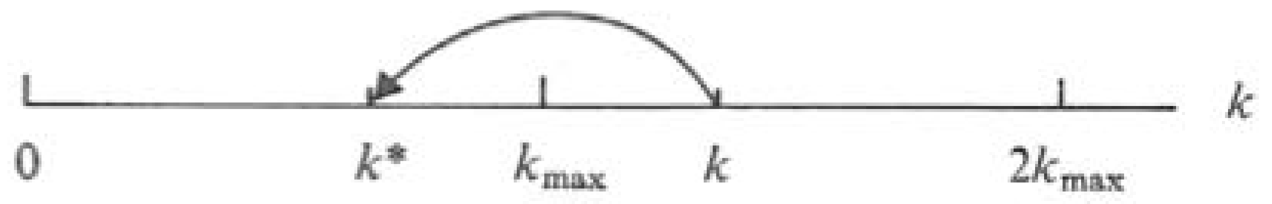


Figure 6.2 Misrepresentation of a wavenumber $k > k_{max}$, in accordance with (6.4).

"Aliasing error"

- In the model, this aliasing is seen as a steady build up of energy in the short portion of the spectrum, and is referred to as "nonlinear instability"

- One solution is to systematically filter out or remove waves that $k > 2/3k_{max}$

- A more elegant solution was suggested by Arakawa, following studies of 2D turbulence in a dynamically closed system

- Charney (1966) pointed out that kinetic energy in such a system can be expressed as

$$\overline{K}\lambda^2 = \frac{1}{2}\overline{\zeta^2} = \sum_n \lambda_n^2 K_n = \text{const}$$

where K=$(u^2+v^2)/2$ and $\zeta^2/2$ are kinetic energy and enstrophy, respectively, and λ^2 is an average square number.

- Fjørtoft (1953) observed that the fraction of the energy that in such a system can flow to high wavenumbers is clearly limited, and the higher the wavenumber, the more it is limited

Technically, if a numerical scheme is designed so as to conserve energy and enstrophy, the energy cascade in the model will avoid the nonlinear instability!!!

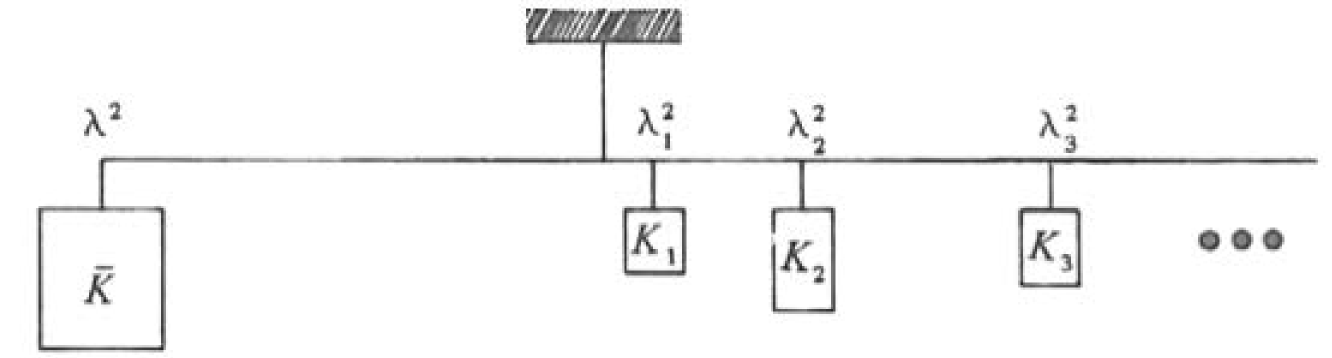


Figure 7.1 A mechanical analogy of the interchange of energy between harmonic components

Ragnar Fjørtoft (1913-1998)

Jule Charney (1917-1981)

Fundamental philosophy of the Belgrade numerical school

- Following principles emphasized by Akio Arakawa:

Design schemes so as to emulate as much as possible physically important features of the continuous system !	Solve the issues by trying to understand them and act on the source of the problem, not on its manifestation

resulted in an unprecedented blossom of the "Belgrade numerical school" :

- Limited Area Primitive Equations Model (LAPEM)
- Hidrometeorological Institute and Belgrade University (HIBU)
- Eta model
- Global Eta Framework (GEF)
- Non-hydrostatic Multiscale Model on B-grid (NMMB)
- DG-Eta ??????

Jankovic, V. (2004): Science migrations: Mesoscale weather prediction from Belgrade to Washington, 1970-2000. *Soc. Stud. Sci.*, **34,** 45-75.

Semi-Lagrangian Schemes

- Eulerian paradigm, where variables are updated at fixed location in space, was dominant over the Lagrangian paradigm, where variables are updated following air parcels

- The situation changed with introduction of semi-Lagrangian approach, where the Lagrangian advection is followed by an immediate remapping back to the grid fixed in space

- Realization that the semi-Lagrangian schemes allow larger time-step than that dictated by the CFL condition, in combination with application of a semi-implicit treatment of gravity-inertial terms, lead to a rapid development of that method

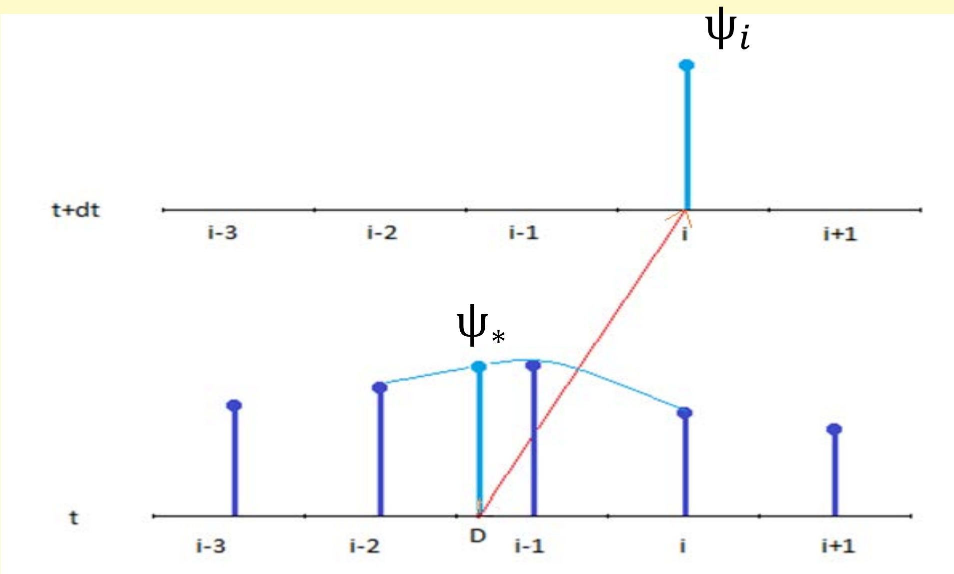
- A detailed review of that development is found in:

> Staniforth, A., & Côté, J. (1991). Semi-Lagrangian integration schemes for atmospheric models – A review. *Monthly Weather Review,* 119(9): 2206–2223.

- We will here touch a few developments that happened in the mean time

Standard SL approach

- Standard, no conservative, method for the semi-Lagrangian advection:



$$\psi_i^{t+dt} = \psi_*^t$$

At current time level (t), we calculate the departure point (D) of a trajectory that will finish at an arrival grid point (i) at next time level (t+dt). Using interpolation from the surrounding points, we estimate the value of the variable at the departure point (ψ_*), and assign that value to the value at the arrival point at the next time level.

- Using a larger time step clearly leads to a larger truncation error in time. However, the theory was that the truncation error in time is much smaller than the truncation error in space, and the method worked fine in practice, even at convective, nonhydrostatic scales

- Two problems posed by the semi-Lagrangian advection were:

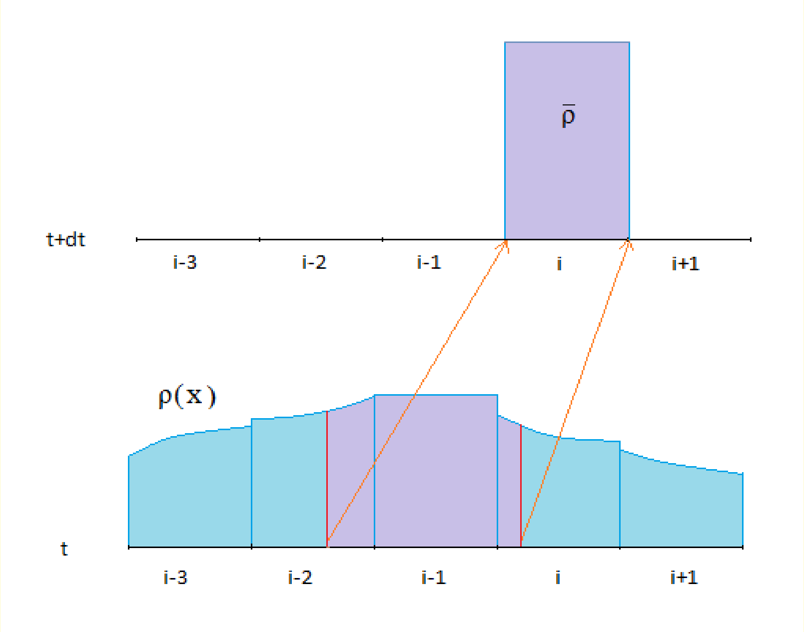
  - A so-called "Gibbs phenomena" (spurious overshoots and undershoots in the solution)
  - Lack of conservation

- The issue of Gibbs oscillations (or monotonization) was addressed in the form of shape-preserving semi-Lagrangian schemes by Williamson and Rash (1989), Rash and Williamson (1990), Skamorock (2006), etc.

- Conservation was considered in the form of a posteriori and a priori methods

- Within a posteriori method the result of advection is manipulated after advection in order to restore the original total values of the advected variable (e.g., Navon 1981, Takacs 1988, Priestley 1993, Sun at al. 1996 and Sun and Sun 2004), including energy restoration (Thuburn et al. 2014; Kent et al. 2016)

A priori conservative SL approach

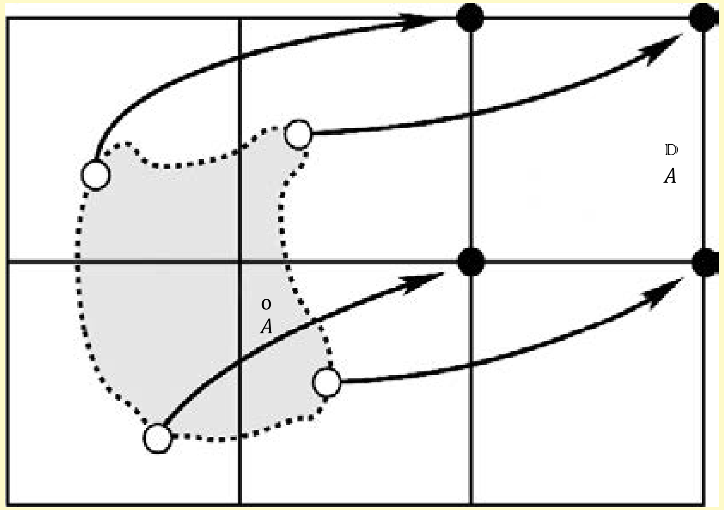


$$\bar{\rho}_i^{t+dt} = \sum \int \rho(x)dx$$

Given the average densities ($\bar{\rho}$) over grid boxes at the initial time level, we first assume a piecewise distribution of density across domain, $\rho(x)$, then cast backward trajectories from time level t+dt to level t, and finally calculate mass between departure points that will arrive into a box i at the next time level, comprising the new average density in the arrival grid box. (Rancic 1992, as an extension of PPM of Colella and Woodward, 1984)

A two-dimensional extension

Trajectories that end at the nodes of an Eulerian grid, form a Lagrangian grid at the previous time level, and the problem is reduced to conservative remapping between the Lagrangian and the Eulerian grids. (Rancic 1992; Machenhauer and Olk. 1998).



Using the Green's theorem, a surface integral is replaced by the summation of the line integrals along edges of the Lagrangian grid box

Machenhauer, B., Kaas, E., & Lauritzen, P. H. (2009). Finite volume techniques in atmospheric models. In: Ciarlet, P., Temam, R., Tribbia, J. (Eds.), *Handbook of numerical analysis: Special volume on computational methods for the atmosphere and oceans*, Elsevier, 14: 3–120, 784 pp.

- An alternative, more efficient method for semi-Lagrangian conservation can be arranged following a cascade interpolation of Purser and Leslie (1991) (Rancic 1995; Leslie and Purser 1995; Zerroukat et al. 2002, 2005, 2006 etc.).

- Potential vorticity, being conserved following air parcel, is considered for a basic quantity of the atmospheric dynamics. SL models based on advection of potential vorticity have been considered (Thuburn 1997; Li at al. 2000). While beyond doubt of high merit, this approach has a serious practical problem of efficiently restoring wind components unsolved.

- A vertically semi-Lagrangian method, where a SL advection is used in vertical, was introduced in Mesinger and Jović (2002)

- The idea of a quasi-Lagrangian vertical coordinate, which originates from Starr (1945), is rediscovered and followed by Kavcic and Thuburn (2018), and is used in FV3 model (Lin 2004; Chen 2013).

Final comments on this topics

- It appears that as spectral, the SL methods also slowly loose ground, primarily due to application of semi-implicit calculation of gravity-inertial terms which involves global operators, inconvenient on the massively parallel machines

- The SL methods are still very much present (or should be present) in application to advection of tracers, as well for vertical advection in atmospheric models

- The issue that still was not sufficiently investigated within SL methods is Lagrangian physics (following a particle of air)

- Finite-differencing methods have been extended to global framework through application of quasi-uniform gridding of the sphere

- However, they are also being replaced by the finite-volume and discontinuous Galerkin methods, which appear to provide much better local control of the solution at very high global resolutions beyond 10km, enabled by the contemporary computers

Representation of spherical Earth

- With the immanent demise of methods based on the spectral transform, and due to problems that convergence of meridians toward poles on the standard spherical grid introduces to local methods, the new topologies for casting grid points over the globe become subject of intensive research

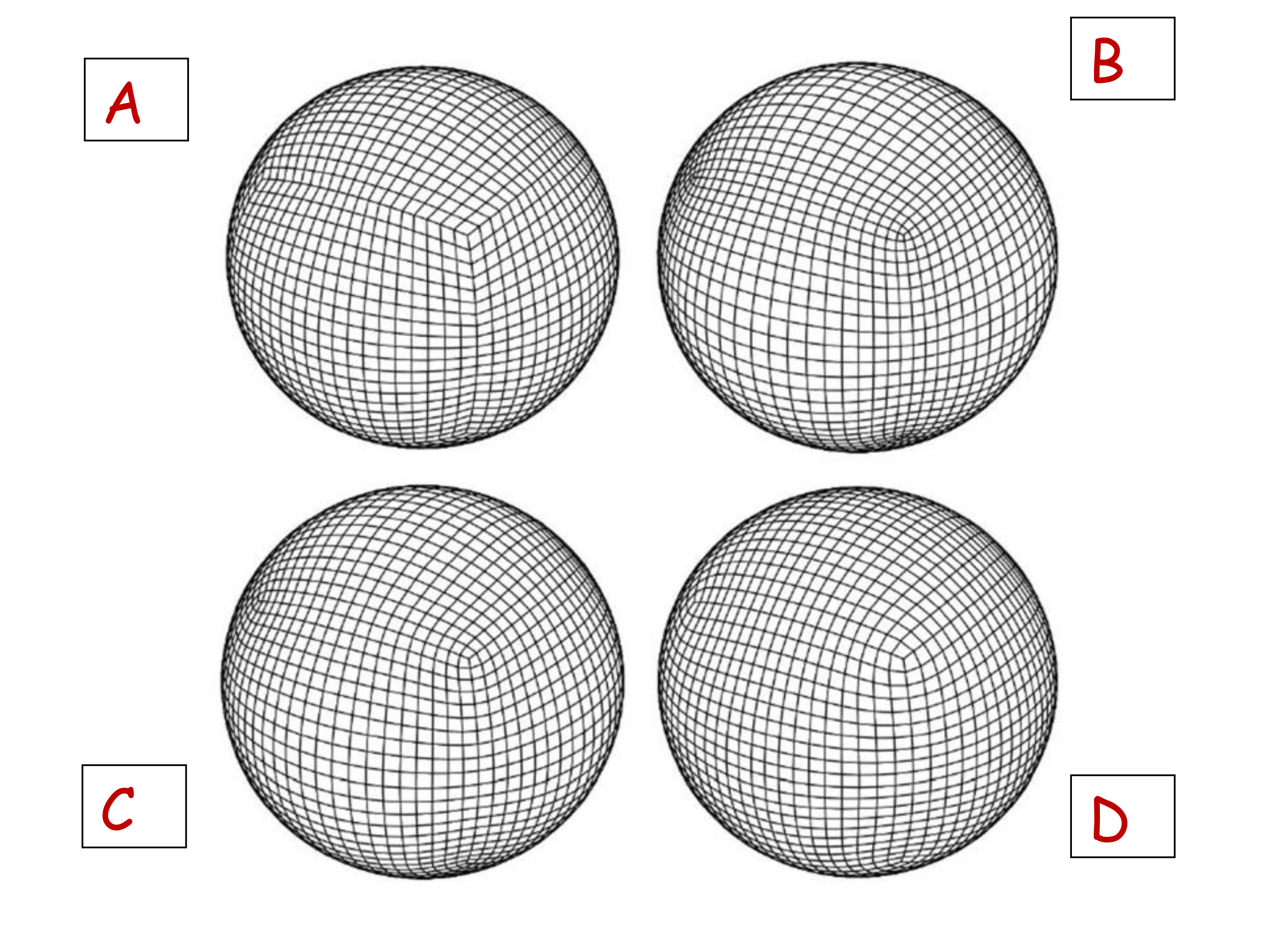
- Comprehensive review of this subject can be found, for example, in:

> Staniforth, A., & Thuburn, J. (2012). Horizontal grids for global weather and climate prediction models: a review. *Quarterly Journal of the Royal Meteorological Society,* 138(662): 1–26. doi:10.1002/qj.958

- Generally, the new spherical topologies require use of general curvilinear coordinates, which brings a series of new issues (formulation of schemes, covariant and contravariant wind components, grid imprints, etc.)

Cubed-sphere

- A – gnomonic (Sadourny 1972)
- B – conformal (first numerically generated)
  (Rancic, Purser, Mesinger 1996)
- C – smoothed
  (Purser and Rancic 1998)
- D – Uniform Jacobian (UJ)
  (Rancic et al. 2017)



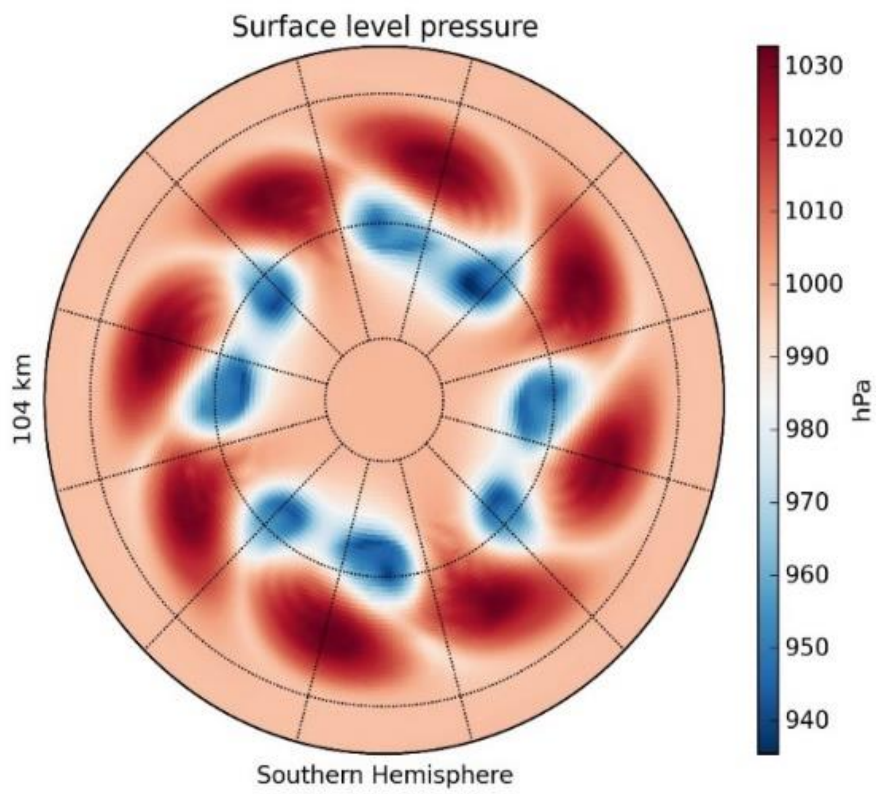
Recently, EMC accepted FV3, a finite-volume model operating on a cubed sphere as the main fore-casting instrument (developed at GFDL by SJ Lin and collaborators)

- Gnomonic cubed-sphere is built by “big circles” (spherical analogies to straight lines) but introduces breaking of coordinate lines across edges

- Cube-spheres with numerically generated coordinates replace that problem with a large curvature of the coordinates close to vertices

- All cube-spheres have 8 weak singular points, and a steady state solution will pass through different Courant numbers

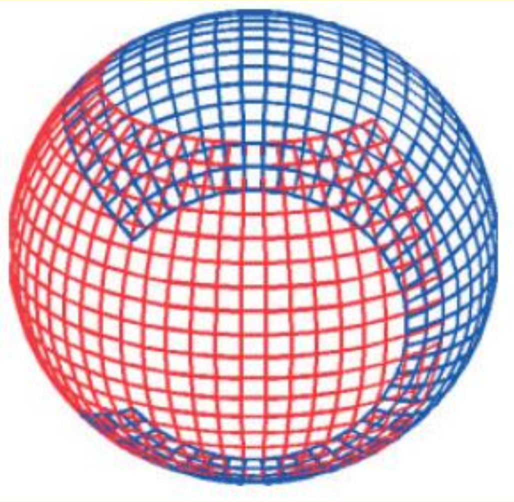
- An undesirable feature of the cubed-sphere is a so-called grid imprint, for which we still do not have a satisfying solution



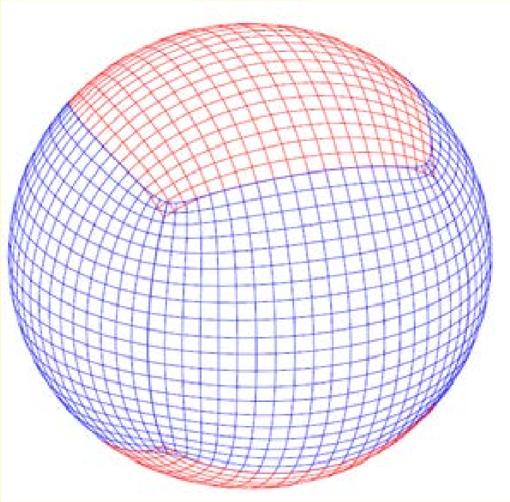
Surface level pressure in the baroclinic instability test case with the NMMB model on UJ cubed-sphere after 17 days of integration (from Rančić et al., 2017).

Overlapping grids

Overlapping (or oversetting, Chimera, etc.) grids avoid the need for curvilinear formulation, but instead require special treatment in the overlap regions, and generally do not automatically satisfy conservation constraints



Yin-yang grid
(Staniforth and Thuburn 2012



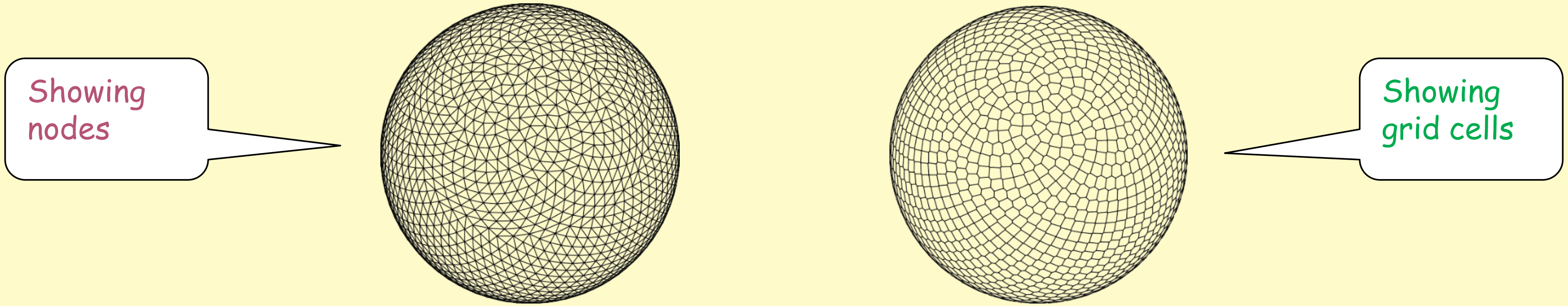
Overset conformal grid
(Purser 2017)

Unstructured grids

- Unstructured grids were until recently only used in oceanography (e.g., Chen et al. 2006; Gao et al. 2011), because of their ability to capture see-land boundaries

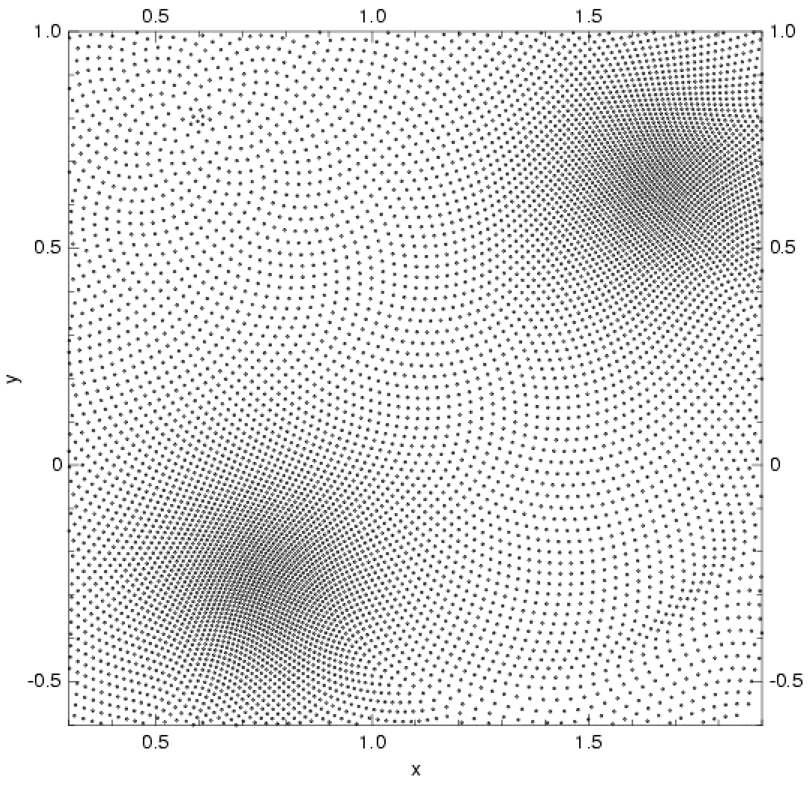
- ECMWF is developing an unstructured grid component (e.g., Smolarkiewicz et al. 2015) because of its grid adapting capability

- We show here a semi-unstructured Fibonacci grid that combines a natural coordinate smoothness with many properties of fully unstructured grids



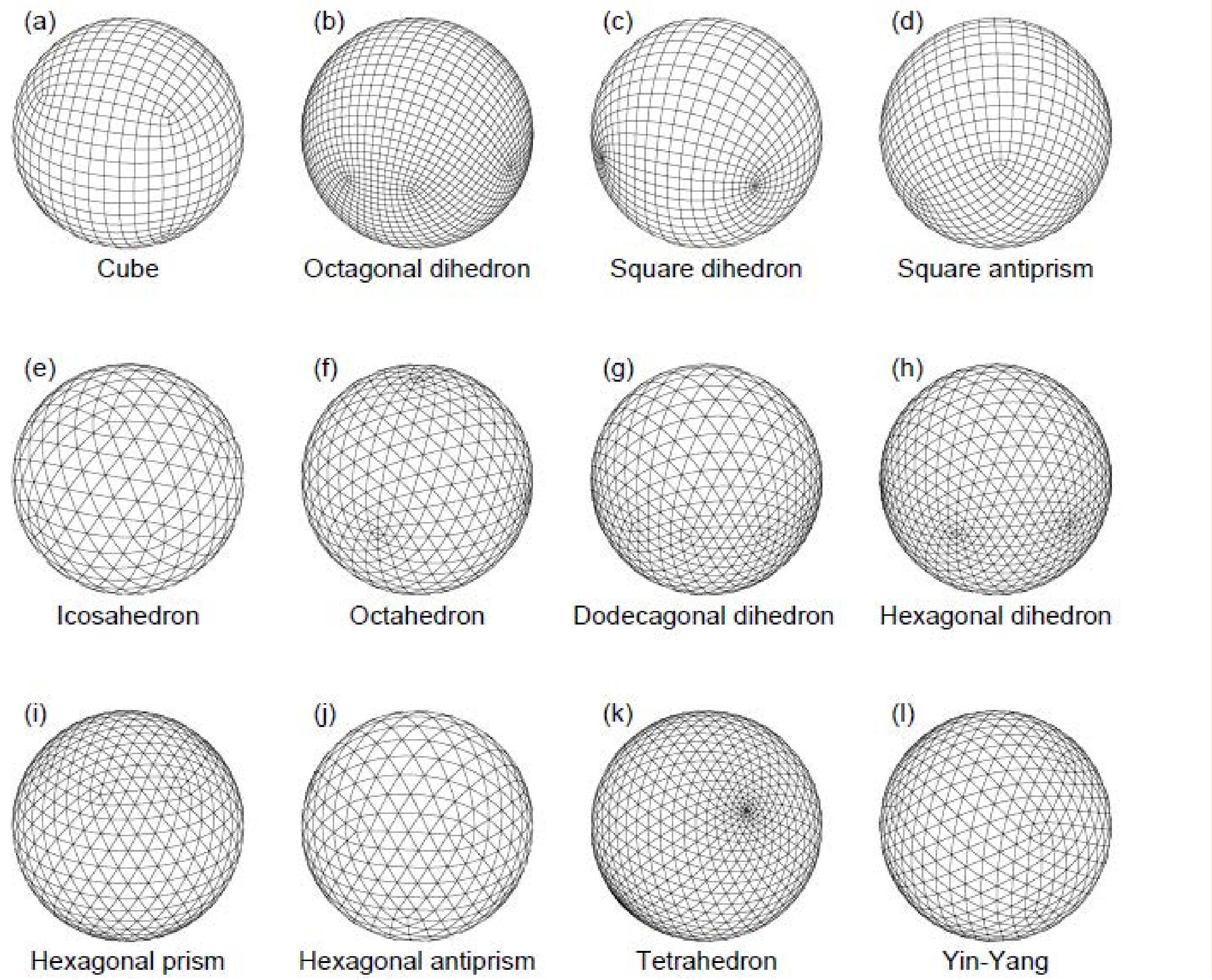
Delaunay triangulation (left) and Voronoi mesh (right) for a Fibonacci grid. (From Swinbank and Purser, 2006).

Grid adaptation with Fibonacci grid



A generalized Fibonacci grid with two domains with locally enhanced resolution (from Purser, 2008).

… and many any others



From Purser and Rancic (2011)

Representation in vertical and topography

Vertical grids

Lorenz (L): Keeps winds and temperature at same level

Charney-Phillips (CP): Vertically splits winds and temperature

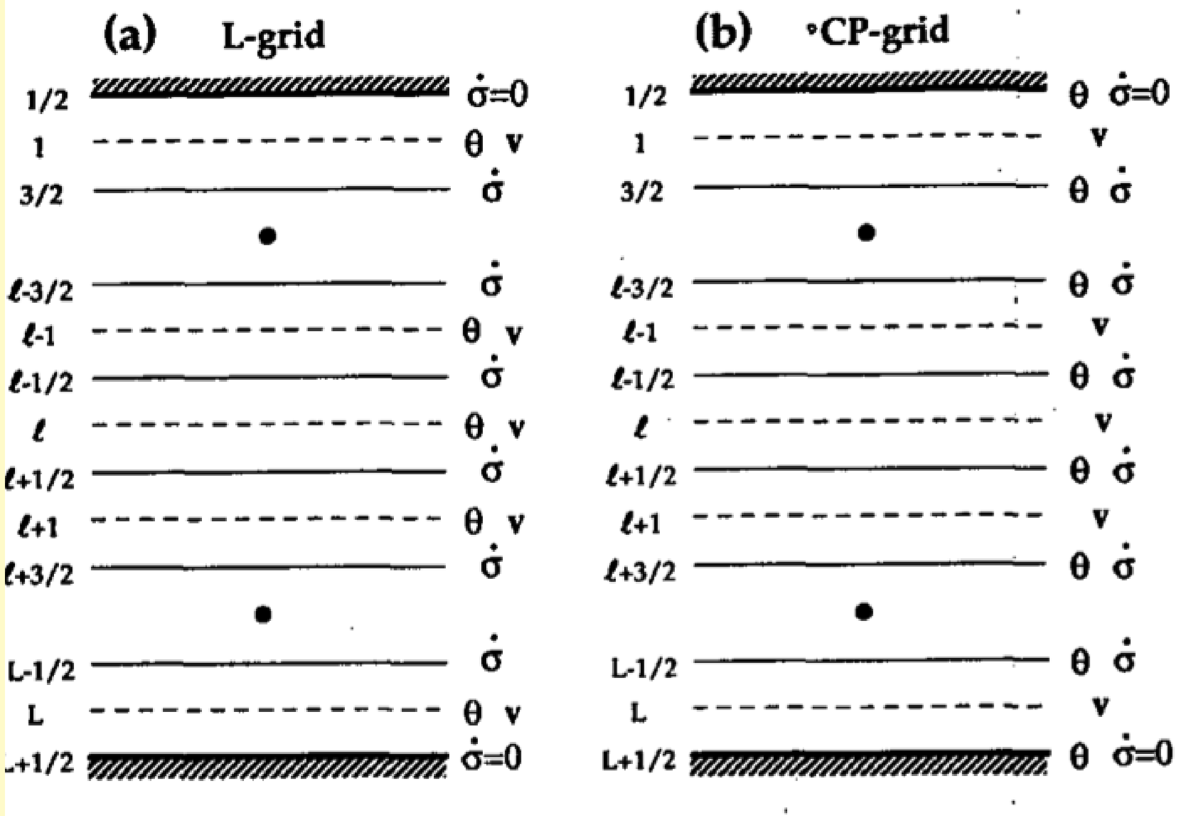


FIG. 1. An illustration of (a) the Lorenz grid and (b) the Charney–Phillips grid for a σ coordinate.

From: Arakawa, A., and C. S. Konor, 1996: *Mon. Wea. Rev.*, **124**, 511-528.

Lorenz grid has a problem with computational mode !!!!!!

Hydrostatic equation: $$\frac{\partial \phi}{\partial \Pi} = -\theta \qquad \Pi = c_p \left(\frac{p}{p_0} \right)^{R/c_p}$$

ϕ - geopotential

Π - Exner function

θ - potential temperature

L:

$$\phi_{l-1} - \phi_l = -\frac{1}{2}\left(\theta_{l-1} + \theta_l\right)\left(\Pi_{l-1} - \Pi_l\right)$$

$$= -\frac{1}{2}\left(\theta'_{l-1} + \theta'_l\right)\left(\Pi_{l-1} - \Pi_l\right)$$

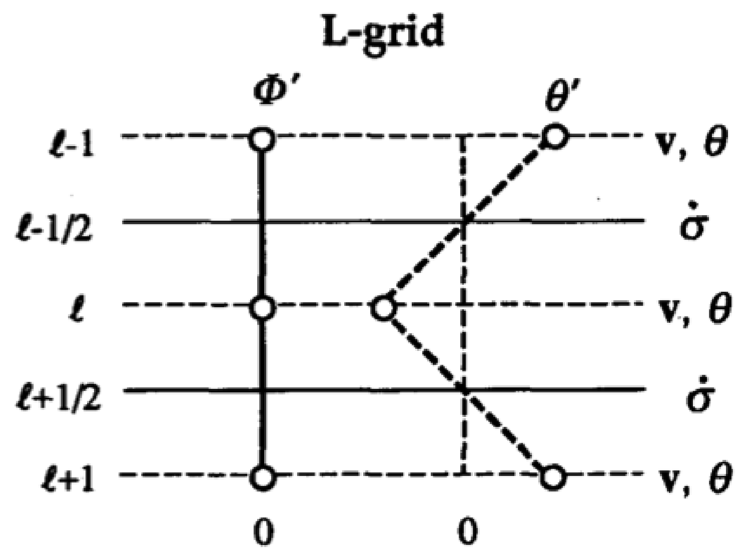


FIG. 2. An illustration of the computational mode on a portion of the L grid.

Arakawa and Konor, 1996: "A zigzag perturbation of potential temperature could be completely decoupled from the dynamics of the discrete system"

CP:

$$\phi_{l-1} - \phi_l = -\theta_{l-1/2}\left(\Pi_{l-1} - \Pi_l\right)$$

- Another problem with Lorenz grid is a spurious amplification of short waves due to barocilinic instability (Arakawa and Moorthy, 1988).

- According to John Thuburn (2011):

  CP: Has generally better wave dispersion properties

  L: Conservation properties are easer satisfied

- The debate between L and CP grid is still going on

- Still, it appears that most models nowadays still prefer L to CP grid!!!!

Vertical coordinates

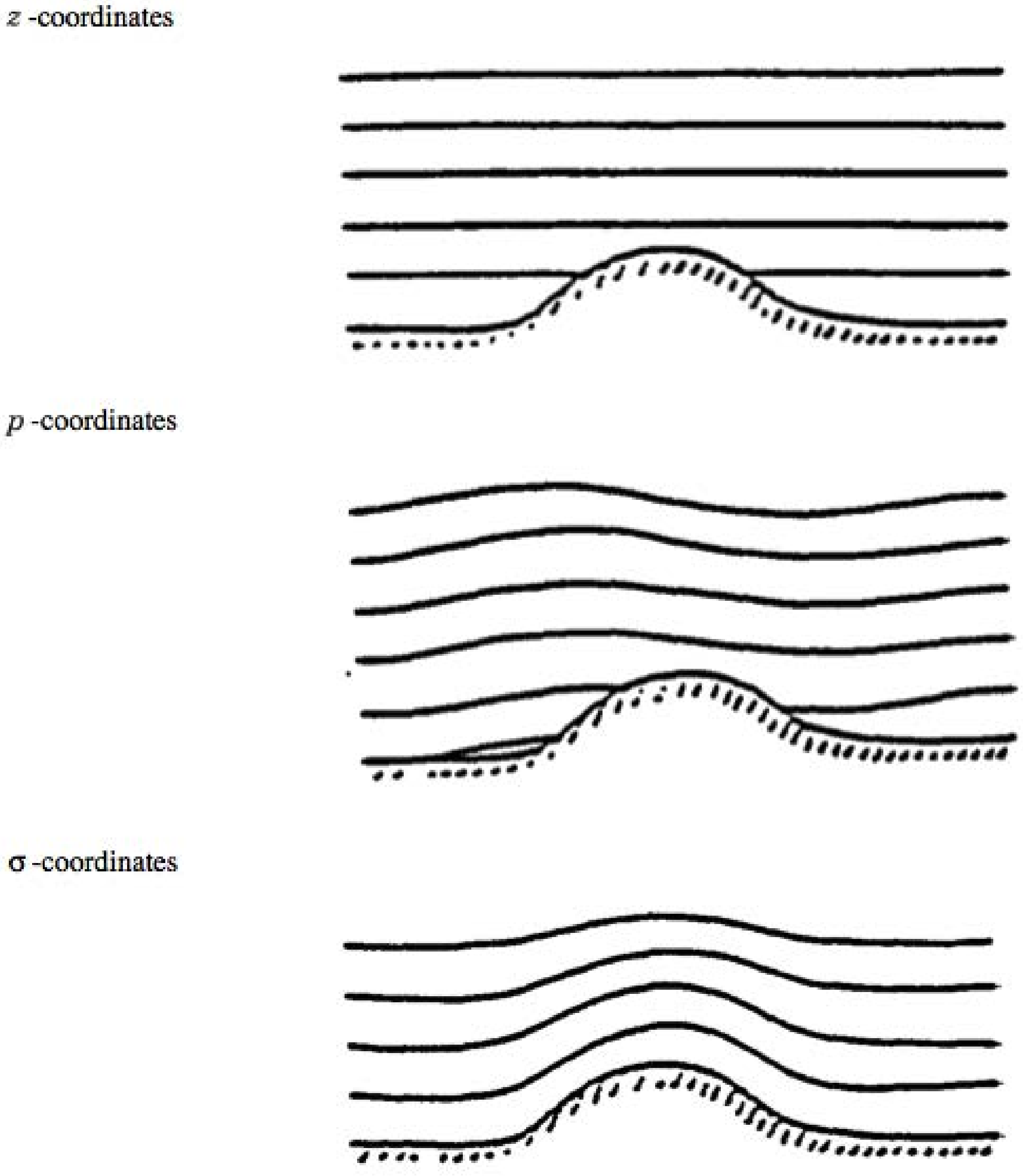


Figure 3Illustration of the three vertical coordinate systems.

z-system: Richardson (1922) first experiment; we measure in troposphere in reference to pressure (maybe for deep "whole atmosphere"?)

p-system: Primitive equations are simple but it intersects topography

σ-system: Norman Philips (1957): $s = \frac{p}{p_S}$ (Or, later, $s = \frac{p - p_T}{p_S - p_T}$) (Arakawa ?)

η-system: Fedor Mesinger (1984): $h = \frac{p - p_T}{p_S - p_T} h_S, \quad h_S = \frac{p_{rf}(z_S) - p_T}{p_{rf}(0) - p_T}$

hybrid (σ/p): Simmons and Burridge (1981); Eckermann (2008):

$$\pi(x,y,s,t) = \pi_T + \sigma_1(s)\Pi + \sigma_2(s)\mu(x,y,t)$$

(NMMB formulation)

Π Constant depth of hydrostatic pressure layer at the top

σ_1 Zero at top and bottom of model atmosphere

σ_2 Increases from 0 to 1 from top to bottom

$\mu = \pi_{Sfc} - \pi_T$ Difference between hydrostatic pressures (π) at surface and top

Problem of the pressure gradient force in sigma system

$$-\nabla_\sigma \phi - \frac{RT}{p}\nabla_\sigma p$$

It is made of two terms that in the presence of steep topography may both become large and create large error. Eta coordinate eliminates the problem by using only quasi-horizontal coordinate surfaces

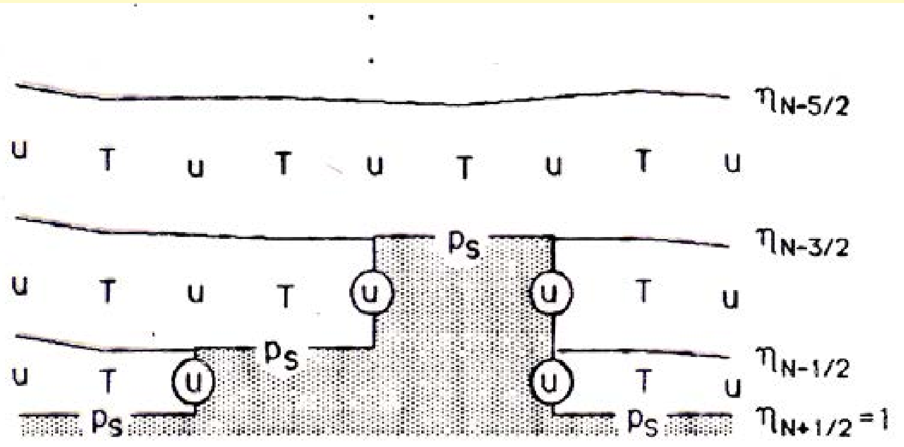


FIG. 1. Schematic representation of a vertical cross section in the eta coordinate using step-like representation of mountains. Symbols u, T and p_s represent the u component of velocity, temperature and surface pressure, respectively. N is the maximum number of the eta layers. The step-mountains are indicated by shading.

Pros:

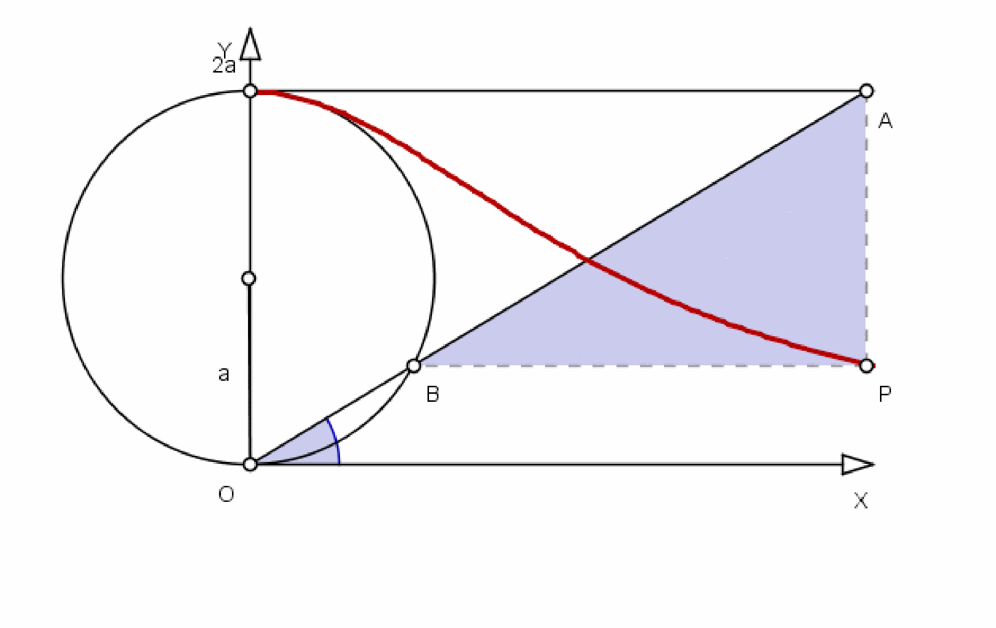
- Blocking effect: Air prefers to move around rather then over the terrain
  Good precipitation scores!!!!!

Cons:

- Formulation of PBL over high terrain
- Internal boundaries inside the domain (4th order schemes; SL advection, …)

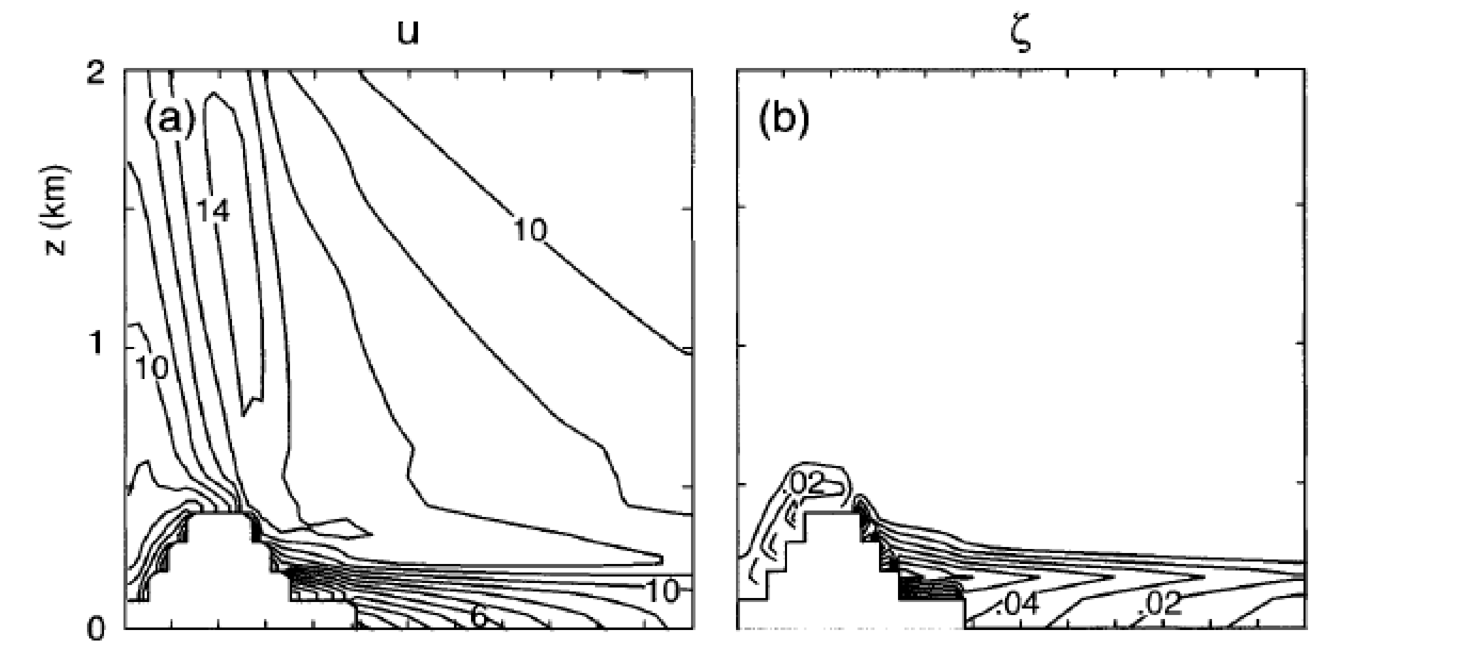
The most serious problem with steep-terrain:

"Witch of Agnesi" – is a curve studied by Maria Agnesi (1748) – the first surviving math work written by a women (according to Wolfram MathWorld) – which loosely describes the flow of air over the sloped terrain



Gallus and Klemp (2000), in simulation of flow over idealized terrain ("Witch of Agnesi"), with a non-hydrostatic version of Eta model (Gallus and Rancic, 1966), found that step-wise terrain leads to artificial separation of flow in the lee of mountain, and attributed that to a spurious generation of vorticity on the steps

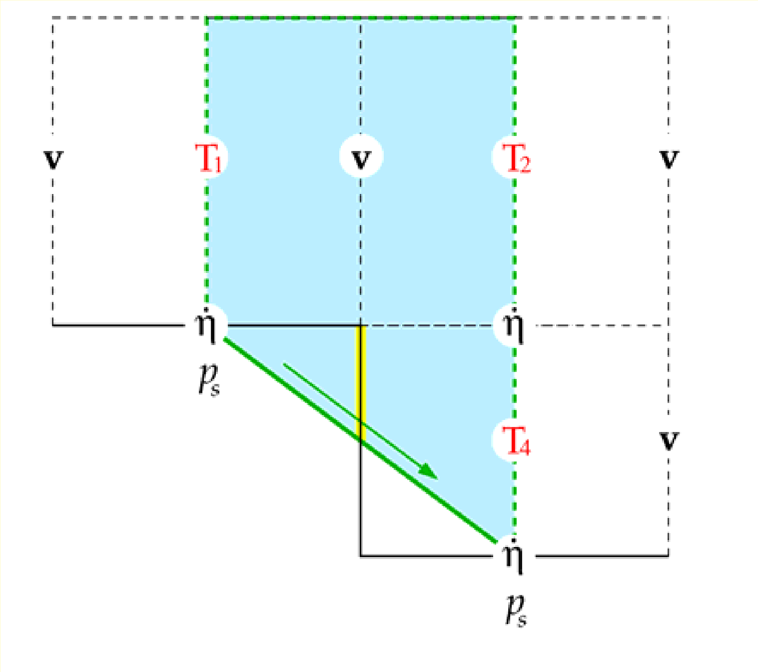
Gallus (2000) confirmed these results in tests with real situations

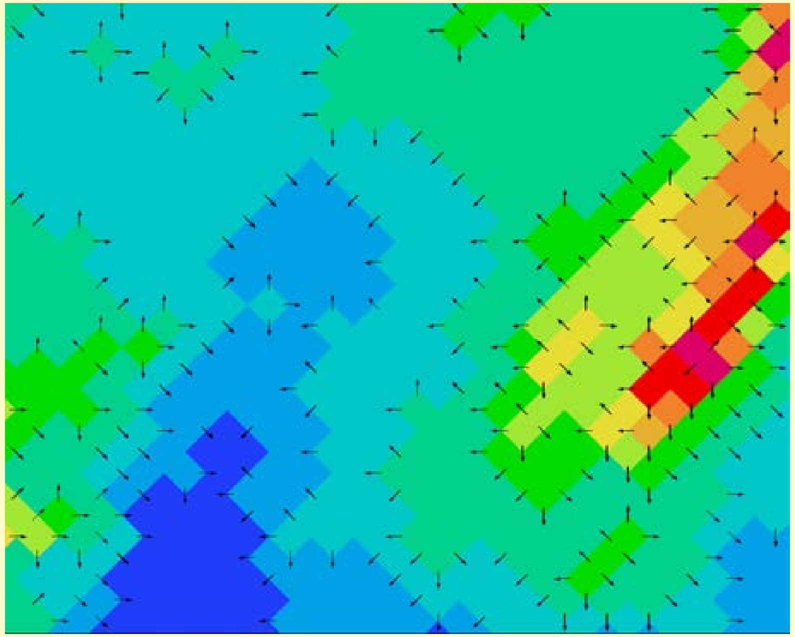


From Gallus and Klemp, MWR, (2000), Fig 6 a and b

Remedy: The sloping steps (Mesinger and Jovic, NCEP Office Note 439 2002)

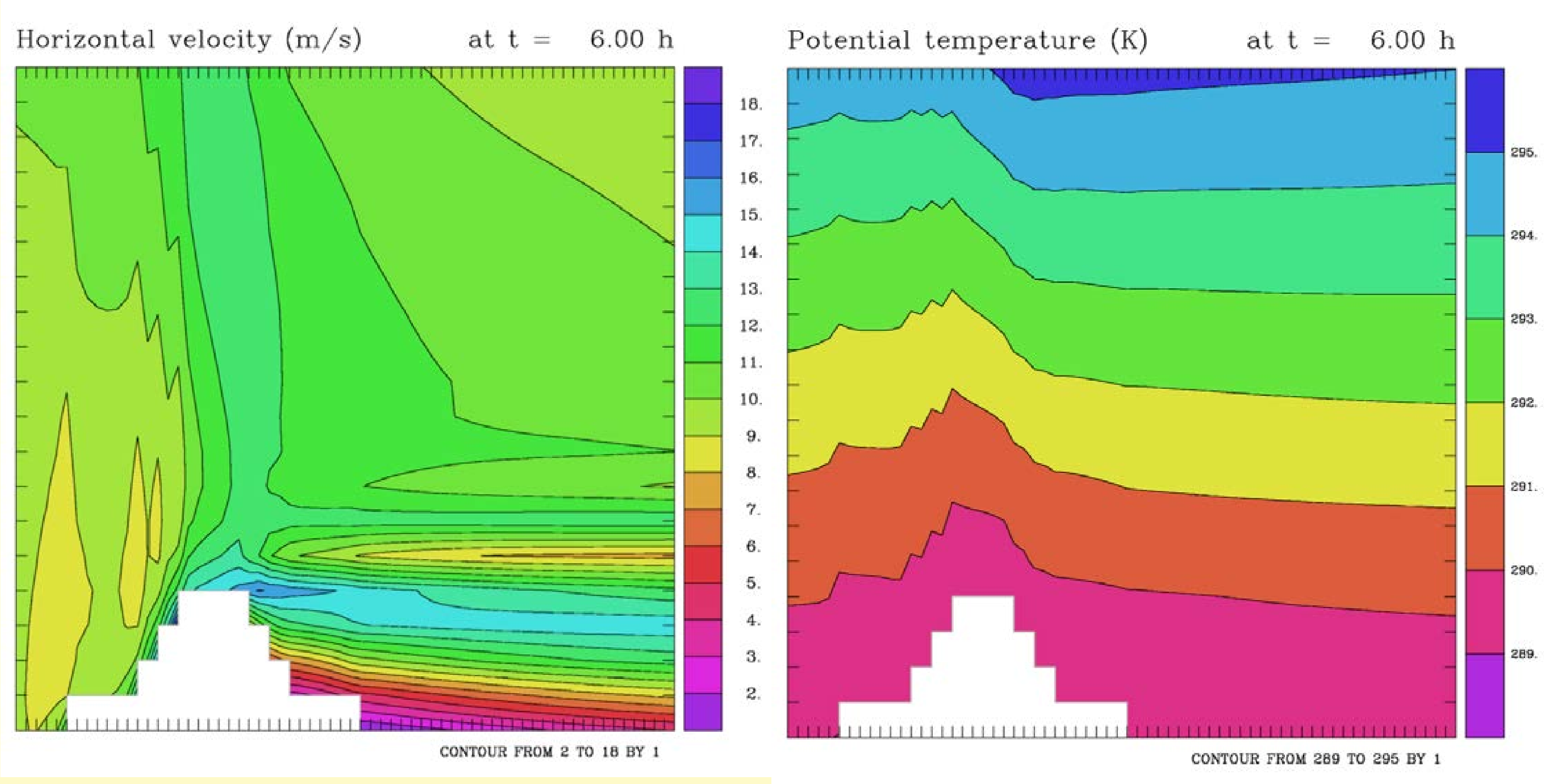
The central **v** box exchanges momentum, on its right side, with **v** boxes of two layers:





Before: The Eta Gallus-Klemp Problem

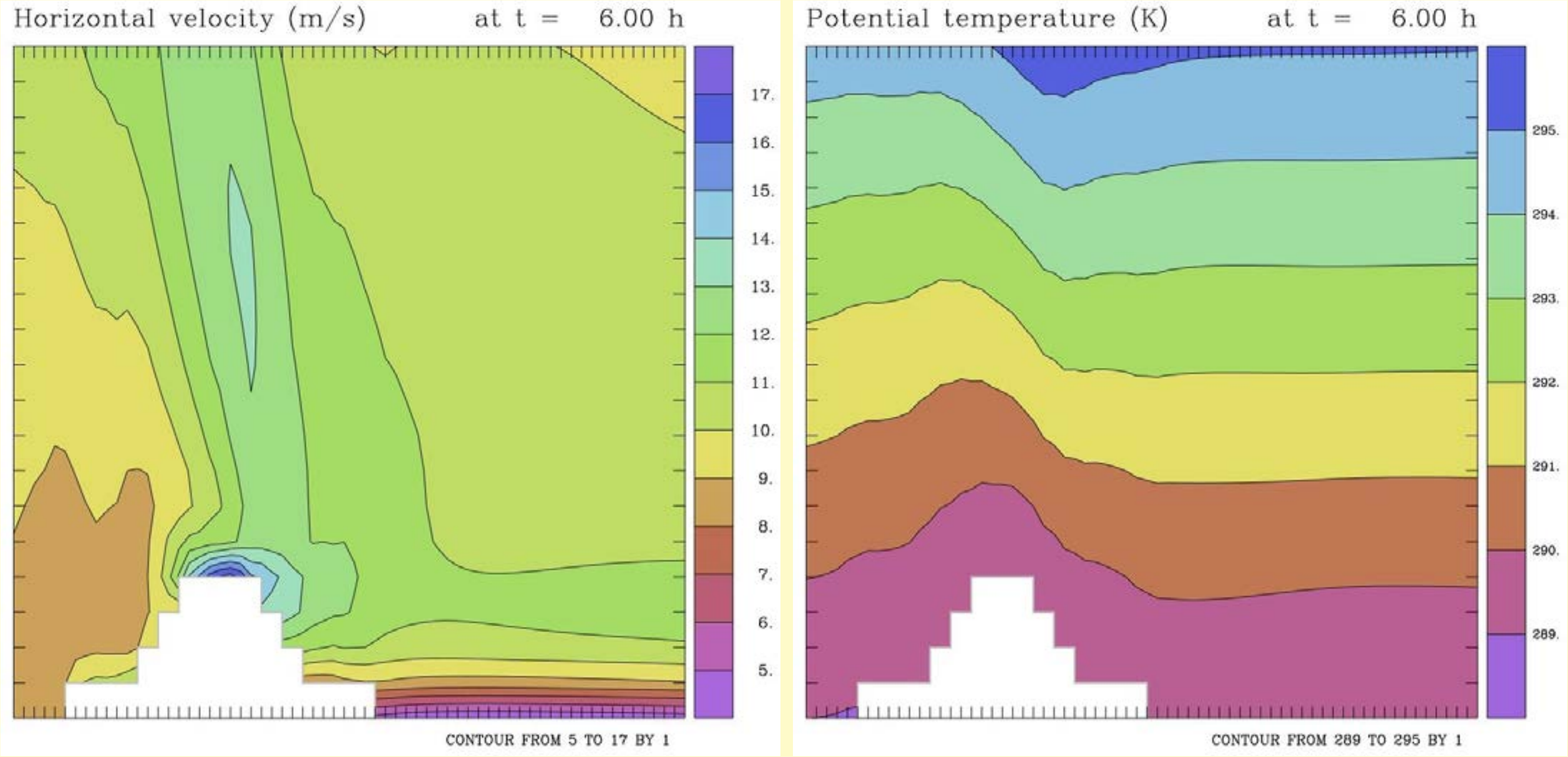
Flow separation on the lee side (à la Gallus and Klemp 2000)



(Hydrostatic; should be nonhydrostatic)

After: Emulation of the Gallus-Klemp experiment, sloping steps code ("poor-man's shaved cells") Unfortunately, run hydrostatic;

left plot from the published paper:



Velocity at the ground immediately behind the mountain increased from between 1 and 2, to between 4 and 5 m/s. "lee-slope separation" much reduced. Zig-zag features in isentropes at the upslope side removed.

- Is it possible that the problem of pressure gradient force of sigma system can be to some degree overcome within finite-volume approach (S.-J. Lin 1997, QJ) ?

- It appears that the main advantage of eta comes from the blocking effect, regardless of issues with the downslope flow

- To some extent, this low level blocking can be achieved in sigma through orographic drag parameterization. A review of that subject can be found in

> Stensrud, D. J. (2009): Parameterization Schemes: Keys to Understanding Numerical Weather Prediction Models, Cambridge University Press, 478 pp

- Mesinger and Veljovic (2017a, b) demonstrated noticeable advantages of eta coordinate in ensemble runs and regional climate simulations, even in comparison with ECMWF. Eta was especially successful in "placement accuracy of upper-air troughs as they move across the Rockies" and precipitation scores

- The "sloped eta" is a simplified version of "shaved or cut-cell terrain", that is receiving a lot of attention lately (Adcroft et al. 1997; Steppeler et al. 2002; Yamazaki and Satomura 2008; Steppeler et al. 2013; Shaw and Weller 2016)

Nonhydrostatic models and NGGPS

- In the early days of weather forecasting, relative shallowness of atmosphere (7 km around poles - 20 km in tropics) led to a series of approximations of Navier-Stokes equations:
  - Hydrostatic: Replaces 3rd equation of motion with the hydrostatic balance equation. Eliminates vertical sound waves, leaving horizontal sound (or Lamb) waves
  - Shallow: Radius of Earth (a) replaces radial distance (R)
  - Traditional: Phillips (1966) suggested that several inertial terms have to be further dropped in order to provide conservation of angular momentum
- Models used for study of mesoscale phenomena and cloud dynamics, where vertical motions are of essence, were using Boussinesq approximation ($\nabla \cdot \boldsymbol{V} = 0$) or anelastic ($\nabla \cdot \overline{\rho}\mathbf{V} = 0$) where $\overline{\rho} = \overline{\rho}(z)$ is some reference density (Ogura and Charny 1962; Ogura and Phillips 1962)

- First steps toward running full compressible models were done by mesoscale (Tapp and White 1976) and cloud (Klemp and Wilhelmson 1978) modelers, who isolated terms responsible for propagation of vertical sound waves and treated them implicitly.

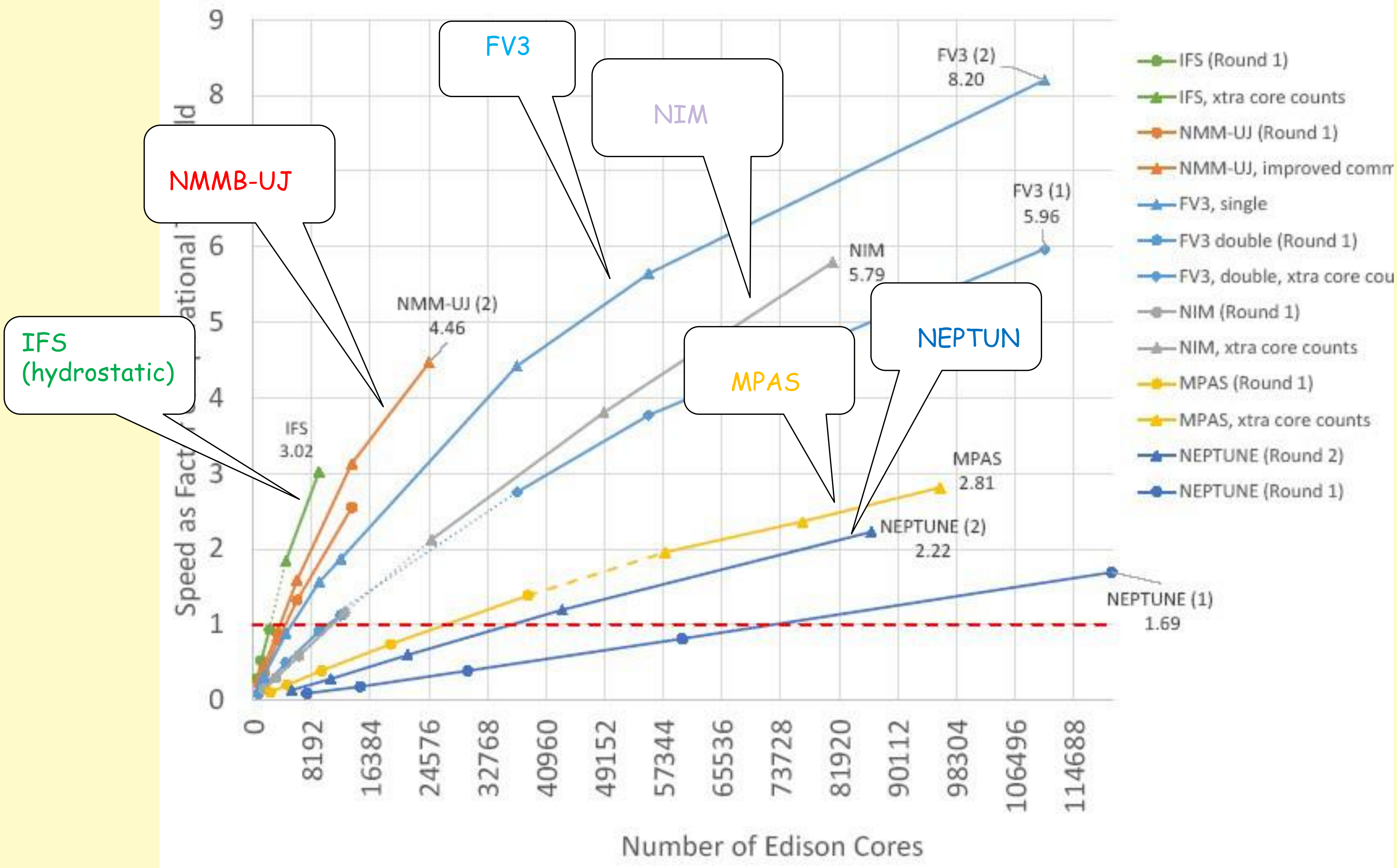
- Most of these models were using geometrical height (z) as the vertical coordinate, which made their application in weather prediction a bit inconvenient

- Laprise (1992) and Juang (1992) independently suggested application of hydrostatic based vertical coordinate in non-hydrostatic models

- Soon, first compressible models based on that approach were developed and tested in operational centers (Bubnova et al. 1995; Gallus and Rancic 1996; Janjic et al. 2001)

- Nowadays, fully compressible models represent standard. In the recently finished US Next Generation Global Prediction System (NGGPS) competition, all competing models were fully compressible:

  - NIM (Nonhydrostatic Icosahedral Model) from NOAA/ESRL (Bleck, et al. 2015)
  - MPAS (Model for Prediction Across Scales) from NCAR (Skamarock et al. 2012)
  - NEPTUN from US NRL (Giraldo et al. 2013)
  - FV3 from NOAA GFDL (e.g., Harris and Lin 2013)
  - NMM-UJ – A UJ version of Nonhydrostatic Multiscale Model from NCEP/NOAA (Rancic et al. 2017)

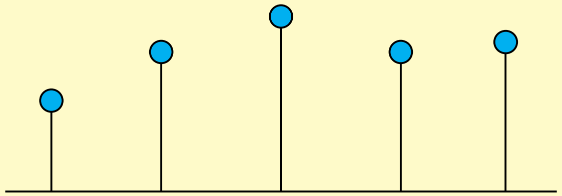


FV3
NIM
NMMB-UJ
IFS (hydrostatic)
MPAS
NEPTUN
FV3 (2) 8.20
FV3 (1) 5.96
NIM 5.79
NMM-UJ (2) 4.46
IFS 3.02
MPAS 2.81
NEPTUNE (2) 2.22
NEPTUNE (1) 1.69
IFS (Round 1)
IFS, xtra core counts
NMM-UJ (Round 1)
NMM-UJ, improved comm
FV3, single
FV3 double (Round 1)
FV3, double, xtra core cou
NIM (Round 1)
NIM, xtra core counts
MPAS (Round 1)
MPAS, xtra core counts
NEPTUNE (Round 2)
NEPTUNE (Round 1)
Number of Edison Cores

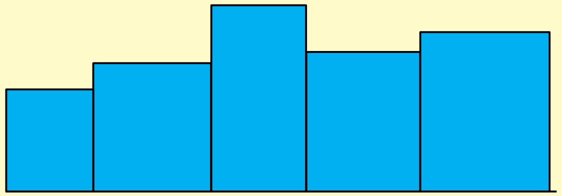
Trends and emerging techniques

Finite volume methods

Unlike finite-difference methods, which deal with the grid-point (nodal) values, finite-volume methods deal with the grid-box averaged values



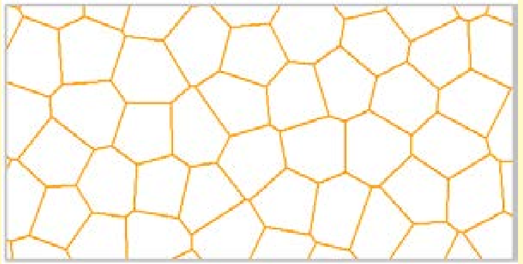
Finite-difference



Finite-volume

Let us consider a mass conservation equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})=0 \qquad \Big/ \qquad \int_S \qquad dS$$



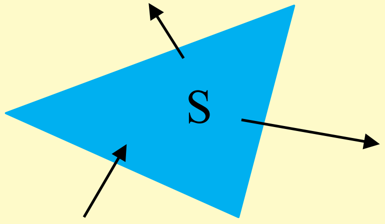
Voronoi Tesselation

which we would like to solve on an arbitrary unstructured grid. We first take integral over area S of a grid cell.

$$\frac{\partial \bar{\rho}}{\partial t} + \int_S \nabla \cdot (\rho \mathbf{V}) dS = 0$$

By applying Gauss theorem, we can convert surface integral into a line integral along boundary of the grid-box

$$\frac{\partial \bar{\rho}}{\partial t} + \oint_L (\rho \mathbf{V}) \cdot \mathbf{n} dl = 0$$



It is convenient to denote fluxes through boundaries with **F**

$$\frac{\partial \bar{\rho}}{\partial t} + \oint_L \mathbf{F} \cdot \mathbf{n} dl = 0$$

In order to update average value in the grid cell, we take integral in time:

$$\bar{\rho}(t+\Delta t) = \bar{\rho}(t) - \int_t^{t+\Delta t} \left(\oint_L \mathbf{F} \cdot \mathbf{n} dl \right) dt$$

Practically, the time change of value in the cell is result of time integral of fluxes across boundaries (Godunov, 1959). An estimation of the time integral of fluxes is referred to as Riemann solver. It can be exact in 1D case but only approximate in 2D case!!!

A comprehensive introduction in the finite-volume method can be found in

LeVeque, R. J. (2002). *Finite Volume Methods for Hyperbolic Problems.* Cambridge Texts in Applied Mathematics. Cambridge University Press, pp 558.

Finite-volume methods easily adjust to quasi-uniform and unstructured grids; they can satisfy most of mimetic constraints, they use only local operators (good for parallelization), but they have low (usually 3rd) order of accuracy.

Two model finalists in the mentioned NGPPS competition (FV3 and MPAS) were finite volume models.

Spectral element methods (e.g. Patera 1984) have a high order of accuracy, but mimetic properties have to be engineered.

Discontinuous Galerkin (DG) method (Reed and Hill 1973; Cockburn et al. 2002) is akin to finite-volume, but has a high-order of accuracy and is inherently conservative. DG carries more information per a control volume, and could be though of as a "high-order finite-volume" method. Examples of modeling of the atmosphere are found in Giraldo et al. (2002); Nair (2009)

(NGGPS participant NEPTUN was a Discontinuous Galerkin model)

Parallelization in time

Though contra intuitive, there is an abundance of literature describing methods for parallelization in time. A review of this subject can be found in:

Gander, M. J. (2015). 50 years of time parallel time integration. In: *Multiple Shooting and Time Domain Decomposition Methods*, MuS-TDD, Heidelberg, May 6–8, 2013, pp 59–113. Springer International Publishing. doi:10.1007/978-3-319-23321-5

The reason to consider these methods is assumption that the number of available processors will increase faster than resolution of the models

In applications to weather prediction, a practical version of this method will probably consist of a series of predictor (parallel) and corrector (sequential) trials.

(An early try in meteorology was done by Côté 2012).

Exponential time integration methods

Capable to accurately describe high frequencies, and still use time steps as large as that of semi-implicit methods. They recently started finding their way into meteorological literature (e.g., Clancy and Pudykiewicz, 2013; Gaudreault and Pudykiewicz, 2016).

Acknowledgement

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- Many thanks to Prof. Mesinger for his effort in organizing this event
- A lot of slides were taken from a joint paper with Dr. Robert James (Jim) Purser

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